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MATRIX ALGEBRAS OF POLYNOMIAL CODIMENSION GROWTH

BY

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ABSTRACT

We study associative algebras with unity of polynomial codimension growth. For any fixed degree k we construct associative algebras whose codimension sequence has the largest and the smallest possible polynomial growth of degree k. We also explicitly describe the identities and the exponential generating functions of these algebras.

1. Introduction: Codimension growth and proper identities

Let A be an associative algebra over a field F and let $c_n(A), n = 1, 2, \ldots$, be its sequence of codimensions. It is well known ([11]) that in case A is a PI-algebra, $c_n(A)$ is exponentially bounded and if $\operatorname{char} F = 0$, either $c_n(A)$ grows exponentially or is polynomially bounded ([8]). In this note we are interested in the case of polynomial growth. For this case it was proved in [2] that $c_n(A)$ behaves asymptotically as

$$c_n(A) = qn^k + \mathcal{O}(n^{k-1}) \approx qn^k, \quad n \to \infty,$$

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for some rational number q. Moreover, if A is a unitary algebra and $k > 1$,

$$\frac{1}{k!} \le q \le \sum_{j=2}^{k} \frac{(-1)^j}{j!} \to \frac{1}{e}, \quad k \to \infty, \tag{1}$$

where $e = 2.71\ldots$ ([4]). In the non-unitary case, for any $0 < q \in \mathbb{Q}$ it is possible to construct an algebra A, such that $c_n(A) \approx qn^k$, for a suitable k ([4]).

The purpose of this note is to construct PI-algebras realizing the smallest and the largest value of q. We shall construct an algebra of upper triangular matrices realizing the value $q = \sum_{j=2}^{k} (-1)^j/j!$. Moreover, we shall prove that the above lower bound (1) is reached only in case k is even. For k odd the lower bound is given by $(k-1)/k!$ and we construct an algebra with such property.

Our technique will be based on the computation of the exponential generating function [7] of the sequence of codimensions and proper codimensions of a PI-algebra.

2. Codimension growth, proper identities and complexity functions

Throughout F is a field and all algebras are F-algebras with 1. Let $F\langle X\rangle$ denote the free associative algebra over F on the countable set $X = \{x_1, x_2, \ldots\}$. We denote by V_n the space of multilinear polynomials in $x_1, \ldots, x_n$, for $n \ge 0$, where we set $V_0 = \mathrm{span}\{1\}$. For a PI-algebra A, we denote by

$$\mathrm{Id}(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ on } A\}$$

the T-ideal of $F\langle X\rangle$ of polynomial identities of A. Recall that

$$c_n(A) = \dim_F \frac{V_n}{V_n \cap \mathrm{Id}(A)}, \quad n = 0, 1, 2, \ldots,$$

is called the sequence of codimensions of A. We define the corresponding complexity function

$$\mathcal{C}(A, z) = \sum_{n=0}^{\infty} \frac{c_n(A)}{n!} z^n,$$

which is the exponential generating function ([7]) of the sequence of codimensions.

A distinguished subspace of V_n is given by the space Γ_n of proper polynomials in $x_1, \ldots, x_n$. Recall that $f(x_1, \ldots, x_n) \in \Gamma_n$ is a proper polynomial if it is a linear combination of products of Lie commutators $[x_{i_1}, \ldots, x_{i_k}]$; we put also $\Gamma_0 = \mathrm{span}\{1\}$. Then one defines the sequence of proper codimensions

$$c_n^p(A) = \dim_F \frac{\Gamma_n}{\Gamma_n \cap \mathrm{Id}(A)}, \quad n = 0, 1, 2, \ldots,$$

and the corresponding exponential generating function

$$\mathcal{C}^p(A,z)=\sum_{n=0}^{\infty}\frac{c_n^p(A)}{n!}z^n.$$

The relation between ordinary and proper codimensions was described by Drensky in [1]. The following result relates the two exponential generating functions.

Lemma 2.1 ([1], [10]): *If A is an associative algebra with identity element, then*

$$\mathcal{C}(A,z)=\exp(z)\cdot\mathcal{C}^p(A,z).$$

Drensky and Regev in [4] proved that if A is a unitary algebra and $c_m^p(A)=0$, for some even integer $m\geq 2$, then $c_n^p(A)=0$ for all $n\geq m$. In case $c_m^p(A)\neq 0$, for every even integer m, it follows that the codimension growth is exponential with exponent at least 2. Therefore, if A is a unitary algebra whose sequence of codimensions is polynomially bounded, $c_m^p(A)=0$ for some even m. It follows that the proper complexity function is a polynomial

$$\mathcal{C}^p(A,z)=\sum_{n=0}^{k}\frac{c_n^p(A)}{n!}z^n,$$

where k is the integer such that $c_k^p(A)\neq 0$ and $c_t^p(A)=0$ for all $t>k$. By Lemma 2.1 from the above relation it follows that the ordinary codimensions are of the form

$$c_n(A)=\sum_{i=0}^{k}\binom{n}{i}c_i^p(A). \tag{2}$$

This is a polynomial in n of degree k, whose leading term is given by $c_k^p(A)\binom{n}{k}$. We therefore obtain the following statement that is also implicitly contained in [4].

Corollary 2.1: *Let A be an associative algebra with identity element. If the codimension sequence $c_n(A)$, $n=0,1,2,\ldots$, is bounded by a polynomial function, then $c_n(A)$ is a polynomial with rational coefficients.*

Suppose now that $c_n(A)=qn^k+\cdots$ is a polynomial of degree k. In [4] it was proved that the leading coefficient q is a rational number satisfying the inequality

$$\frac{1}{k!}\leq q\leq\sum_{j=2}^{k}\frac{(-1)^j}{j!}\to\frac{1}{e},\quad k\to\infty. \tag{3}$$

We first improve the above lower bound for k odd. In fact we have

PROPOSITION 2.1: *Let A be a unitary PI-algebra over a field of characteristic zero. If $c_n(A) = qn^k + \mathcal{O}(n^{k-1})$, for some odd integer $k > 1$ and rational number q, then $q \geq (k-1)/k!$.*

Proof: As we remarked above

$$c_n(A) = \sum_{i=0}^{k} \binom{n}{i} c_i^p(A) = c_k^p(A)\binom{n}{k} + \cdots \approx \frac{c_k^p(A)}{k!} n^k, \quad n \to \infty.$$

Hence $q = c_k^p(A)/k!$ and we need to compute the smallest possible value of $c_k^p(A)$.

It is well known that the symmetric group S_k acts on the vector space $V_k = V_k(x_1, \ldots, x_k)$ by permuting the variables and V_k is isomorphic to the regular S_k-module (see, for instance, [6, Section 2.4]). Therefore, V_k has only two one-dimensional submodules corresponding to the diagrams $\lambda = (k)$ and $\lambda = (1^k)$. Since $k \neq 4$, it is well-known that the dimension of any other irreducible submodule is at least $k-1$. The one-dimensional submodules are spanned by the polynomials

$$f_k = \sum_{\pi \in S_k} x_{\pi(1)} \cdots x_{\pi(k)}$$

and

$$St_k = \sum_{\pi \in S_k} (-1)^\pi x_{\pi(1)} \cdots x_{\pi(k)},$$

respectively. Clearly, $\Gamma_k \subset V_k$ is an S_k-submodule and, since $\Gamma_k \cap \mathrm{Id}(A)$ is invariant under permutations of the variables, $\frac{\Gamma_k}{\Gamma_k \cap \mathrm{Id}(A)}$ becomes an S_k-module. Its character, denoted $\chi_k^p(A)$, is called the proper k-th cocharacter of A. By complete reducibility $\chi_k^p(A)$ decomposes into irreducibles and let

$$\chi_k^p(A) = \sum_{\lambda \vdash k} m_\lambda \chi_\lambda, \tag{4}$$

where χ_λ is the irreducible S_k-character associated to the partition λ and m_λ is the corresponding multiplicity. Thus $c_k^p(A) = \sum_{\lambda \vdash k} m_\lambda \chi_\lambda(1)$.

It can be easily checked (see, for instance, [3, Exercise 4.3.6]) that $f_k \notin \Gamma_k$ for all k. Also, $St_k \notin \Gamma_k$ for odd k and $St_k \in \Gamma_k$ for even k. Hence, when k is odd the lowest possible degree of a character appearing in (4) with non-zero multiplicity must be $k-1$. It follows that $q = c_k^p(A)/k! \geq (k-1)/k!$. ∎

In the next section, by constructing suitable algebras, we shall prove that the upper bound and the lower bound of q are actually reached for every k.

If we apply Lemma 2.1 to the free algebra $F\langle X\rangle$ of countable rank, we obtain a series enumerating the proper associative polynomials

$$\mathcal{C}^p(F\langle X\rangle, z) = \exp(-z)\mathcal{C}(F\langle X\rangle, z) = \frac{\exp(-z)}{1-z} = \left(\sum_{j=0}^{\infty}\frac{(-1)^j z^j}{j!}\right)\left(\sum_{i=0}^{\infty} z^i\right)$$
$$= \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{(-1)^j}{j!}\right)z^i = \sum_{i=0}^{\infty}\theta_i z^i = 1 + \frac{z^2}{2} + \frac{2z^3}{6} + \frac{9z^4}{24} + \cdots,$$

where

$$\theta_i = \sum_{j=0}^{i}\frac{(-1)^j}{j!} = \frac{\dim \Gamma_i}{i!}, \quad \text{for } i \in \mathbb{N}.$$

3. Constructing PI-algebras

In this section, for any fixed $k > 1$, we shall construct associative algebras with unity, whose codimension sequence is given asymptotically by the largest or smallest possible polynomial of degree k.

Let $U_k = U_k(F)$ be the algebra of $k \times k$ upper triangular matrices with equal entries in the main diagonal. Hence if the e_{ij}'s are the usual matrix units and $E = E_{k\times k}$ denotes the identity $k \times k$ matrix,

$$U_k = \left\{\alpha E + \sum_{1\le i<j\le k}\alpha_{ij}e_{ij} \mid \alpha, \alpha_{ij} \in F\right\}.$$

Let also $SU_k = SU_k(F)$ denote the algebra of strictly upper triangular matrices over F. In the next theorem we shall prove that the algebra U_k has the largest possible polynomial growth of degree $k-1$, namely $c_n(U_k) \approx qn^{k-1}$ as $n \to \infty$, where $q = \sum_{j=2}^{k-1}(-1)^j/j!$. In what follows Lie commutators are left-normed, i.e., $[x_1, x_2, \ldots, x_k] = [\cdots[[x_1, x_2], x_3], \ldots, x_k]$.

Theorem 3.1: *Let F be an infinite field. Then:*

1) *A basis of the identities of U_k is given by all products of commutators of total degree k*

$$(5) \qquad [x_1, \ldots, x_{a_1}][x_{a_1+1}, \ldots, x_{a_2}]\cdots[x_{a_{r-1}+1}, \ldots, x_{a_r}]$$

with $a_r = k$ in case k is even, and by the polynomials in (5) plus the polynomial of degree $k+1$

$$[x_1, x_2]\cdots[x_k, x_{k+1}]$$

in case k is odd.

2)

$$\mathcal{C}(U_k, z) = \exp(z) \sum_{i=0}^{k-1} \theta_i z^i.$$

3)

$$c_n(U_k) = \sum_{j=0}^{k-1} \frac{n!}{(n-j)!}\theta_j \approx \theta_{k-1} n^{k-1}, \quad n \to \infty.$$

Proof: For $u_1, \ldots, u_t \in U_k$ write $u_i = \alpha_i E + v_i$ with $v_i \in SU_k$, $1 \le i \le t$. Then

$$[u_1, \ldots, u_t] = [v_1, \ldots, v_t] \in (SU_k)^t,$$

and, since $(SU_k)^k = 0$, all polynomials in (5) yield identities. If k is odd, $[x_1, x_2] \cdots [x_k, x_{k+1}]$ is also an identity of U_k.

Let now $f \in \mathrm{Id}(U_k)$. Since the polynomial identities of a unitary algebra over an infinite field follow from the proper ones [3, Proposition 4.3.3], we may assume that f is proper. Also, it is well known that $x_1 \cdots x_k$ is a basis of the identities of SU_k ([9], [12]); hence, in particular, SU_k does not satisfy any identity of degree less than k. Since f is an identity of $SU_k \subset U_k$, then $\deg f \ge k$. On the other hand, it is clear that a commutator of length $m > 2$ is a consequence of any commutator of length $< m$. Hence a product of commutators of total length $m \ge k$, and so f, follows either from one of the polynomials in (5) or from the polynomial $[x_1, x_2] \cdots [x_k, x_{k+1}]$. This proves the first claim.

From 1) it follows that if $f \in \Gamma_t$, $t < k$, then f is not an identity of U_k. Hence, for any $t < k$, $c_t^p(U_k) = \dim_F \Gamma_t = t!\theta_t$ and we get 2) and 3). ∎

The importance of U_k is shown in the following.

Theorem 3.2: *Let A be a unitary algebra over an infinite field F such that $c_n(A) \approx qn^k$, $n \to \infty$. Then* $\mathrm{Id}(A) \supseteq \mathrm{Id}(U_{k+1})$.

Proof: By (2) we have that $c_n(A) = \binom{n}{k} c_k^p(A) + \cdots$ and $c_{k+i}^p(A) = 0$, $i \ge 1$. This says that $\Gamma_{k+i} = \Gamma_{k+i} \cap \mathrm{Id}(A)$, i.e., $\Gamma_{k+i} \subseteq \mathrm{Id}(A)$, $i \ge 1$. Since by the previous theorem $\mathrm{Id}(U_{k+1})$ is generated by Γ_{k+1}, and in case k is even, also by $[x_1, x_2] \cdots [x_{k+1}, x_{k+2}] \in \Gamma_{k+2}$, we get that $\mathrm{Id}(U_{k+1}) \subseteq \mathrm{Id}(A)$. ∎

We now turn to the problem of constructing algebras of polynomial codimension growth realizing the minimal possible value for q.

Suppose that $k \ge 3$. Let $J = \sum_{i=1}^{k-1} e_{i,i+1} \in U_k$ denote the diagonal just above the main diagonal of U_k. For all $l \in \{1, 2, \ldots, k-1\}$ define subalgebras of U_k as follows:

$$\begin{aligned} N_{k,l} &= N_{k,l}(F) \\ &= \mathrm{span}\{E, J, J^2, \dots, J^{k-l-1}; e_{12}, e_{13}, \dots, e_{1,k-l}; e_{ij} \mid j - i \geq k - l\}. \end{aligned}$$

The following case

$$N_k = N_{k,1} = \mathrm{span}\{E, J, J^2, \dots, J^{k-2}; e_{12}, e_{13}, \dots, e_{1k}\} = N_{k,2}$$

is of special interest for us. We remark that this algebra is generated by $\{E, J, e_{12}\}$. In general, we have the following strict inclusions:

$$N_{k,1} = N_{k,2} \subset \cdots \subset N_{k,k-1} = U_k.$$

We are going to show that the algebras $N_k, N_{k,3}, \dots, N_{k,k-2}$ have asymptotically the same codimension growth which is different from that of $N_{k,k-1} = U_k$ for $k > 4$.

THEOREM 3.3: *Let $1 \leq l \leq k-2$, $k > 4$. If F is an infinite field then:*

1) *$N_{k,l}$ and U_k generate different varieties.*
2)
$$c_n(N_{k,l}) \approx \frac{k-2}{(k-1)!} n^{k-1}, \quad n \to \infty.$$

Proof: Since $N_{k,l}$ is a subalgebra of U_k, $c_n(N_{k,l}) \leq c_n(U_k)$. Hence $c_n^p(N_{k,l}) = 0$ for all $n \geq k$ and we need to compute $c_{k-1}^p(N_{k,l})$. To this end, we claim that the polynomial $[x_1, \dots, x_{k-1}]$ is not an identity of $N_{k,l}$. In fact, by evaluating $x_1 = e_{12}, x_2 = \cdots = x_{k-1} = J$ we get $[e_{12}, J, \dots, J] = e_{1k} \neq 0$. On the other hand, any product of Lie commutators

$$[x_1, \dots, x_{r_1}][x_{r_1+1}, \dots, x_{r_2}] \cdots [x_{r_{t-1}+1}, \dots, x_{r_t}] \tag{6}$$

of total degree $r_t = k - 1$ and containing $t \geq 2$ commutators is an identity of $N_{k,l}$. Indeed, consider the polynomial $[x_1, x_2, \dots, x_r]$, where $r \geq 2$. Then by induction on r, one proves that all its evaluations in elements of $N_{k,l}$ are contained in $\mathrm{span}\{e_{1,r+1}, e_{1,r+2}, \dots; e_{ij} \mid j - i \geq k - l + r - 1\}$. We observe that the first part of this set lies on or above the rth diagonal above the main diagonal while the second part lies on or above the diagonal $r + (k-l-1) \geq r+1$. Now consider the product (6). Only the product of elements from the diagonals $r_1, (r_2 - r_1), \dots, (r_t - r_{t-1})$ can give a non-zero element, since $r_t = k - 1$. But the corresponding elements belong to the first row. Hence, (6) is an identity of $N_{k,l}$, which is not an identity of U_k by Theorem 3.1. We obtain that Γ_{k-1}

modulo $\mathrm{Id}(N_{k,l})$ is spanned by the commutators $[x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}}]$ of length $k-1$. We claim that we may take $i_1 > i_2 \leq \cdots \leq i_{k-1}$. To this end we prove that $N_{k,l}$ satisfies the identities

$$[[x_1, \ldots, x_r], [x_{r+1}, x_{r+2}], x_{r+3}, \ldots, x_{k-1}], \quad r \geq 2. \tag{7}$$

Indeed, by the argument above, any evaluation of the first two commutators lies on or above the rth diagonal and the second diagonal, respectively, while the remaining $k-r-3$ factors lie at least on the first diagonal. Only the extreme case could yield a nonzero product, but in this case the first two factors belong to the first row and we get zero. The identities (7) allow us to permute the elements $x_{i_3}, \ldots, x_{i_{k-1}}$ in the commutator above in an arbitrary way. Also, we can make x_{i_2} minimal among $\{x_{i_1}, x_{i_2}, x_{i_3}\}$ using the Jacobi identity.

Therefore, the polynomials $[x_i, x_1, \ldots, \hat{x}_i, \ldots, x_{k-1}]$, $i = 2, 3, \ldots, k-1$, where the symbol $\hat{x}_i$ means that the variable x_i is omitted, span Γ_{k-1} modulo $\mathrm{Id}(N_{k,l})$. By making the substitutions of Theorem 3.4 below, it follows that they form a basis of the multilinear proper polynomials of degree $k-1$ mod $\mathrm{Id}(N_{k,l})$. Hence $c^p_{k-1}(N_{k,l}) = k-2$ and

$$c_n(N_{k,l}) \approx \binom{n}{k-1} c^p_{k-1}(N_{k,l}) \approx \frac{k-2}{(k-1)!} n^{k-1} \quad \text{as } n \to \infty. \qquad \blacksquare$$

Next we describe explicitly the identities of $N_k = N_{k,1} = N_{k,2}$. Recall that we define $N_{k,l}$ only if $k \geq 3$.

THEOREM 3.4: *Let $k \geq 3$ and let F be an infinite field. Then:*

1) *A basis of the identities of N_k is given by the polynomials*

$$[x_1, \ldots, x_k], \; [x_1, x_2][x_3, x_4]. \tag{8}$$

2)

$$\mathcal{C}(N_k, z) = \exp(z)\left(1 + \sum_{j=2}^{k-1} \frac{j-1}{j!} z^j\right).$$

3)

$$c_n(N_k) = 1 + \sum_{j=2}^{k-1} (j-1)\binom{n}{j} \approx \frac{k-2}{(k-1)!} n^{k-1}, \quad n \to \infty.$$

Proof: It is clear that $[N_k, N_k] \subseteq \mathrm{span}\{e_{13}, e_{14}, \ldots, e_{1k}\}$. Hence $[x_1, x_2][x_3, x_4]$ is an identity of N_k. The other identity in (8) follows from Theorem 3.1 since N_k is a subalgebra of U_k.

Let now f be an identity of N_k. We may clearly assume that f is multilinear, and since N_k is an algebra with 1 we may take f proper. After reducing the polynomial f modulo the identities in (8), by the proof of the previous theorem, we obtain that f can be written as a linear combination of left-normed commutators of length say $s \leq k-1$,

$$f = \sum_{j=2}^{s} \alpha_j [x_j, x_1, x_2, \ldots, \hat{x}_j, \ldots, x_s], \quad \alpha_j \in F.$$

Suppose that $\alpha_2 \neq 0$. We evaluate $x_2 = e_{12}$, $x_1 = J$, and $x_3 = \cdots = x_s = J$ and obtain $f(J, e_{12}, J, \ldots, J) = \alpha_2 [e_{12}, J, \ldots, J] = \alpha_2 e_{1s+1} \neq 0$, a contradiction.

The arguments above also prove that the polynomials $[x_i, x_1, \ldots, \hat{x}_i, \ldots, x_s]$, where $i = 2, 3, \ldots, s$, yield a basis of the multilinear proper polynomials of degree s modulo $\mathrm{Id}(N_k)$. Hence $c_s^p(N_k) = s-1$ for $s = 2, \ldots, k-1$, and we get 2), and 3). ∎

Let now G_{2k} be the Grassmann algebra with unity on a $2k$-dimensional vector space over a field F of characteristic not equal to two. Recall that

$$G_{2k} = \langle 1, e_1, \ldots, e_{2k} \mid e_i e_j = -e_j e_i \rangle.$$

Then $G_{2k} = \mathrm{span}\{e_{i_1} \cdots e_{i_r} \mid 0 \leq i_1 < \cdots < i_r \leq 2k\}$ has a natural $\mathbb{Z}_2$-grading $G_{2k} = G_{2k}^{(0)} \oplus G_{2k}^{(1)}$ where $G_{2k}^{(0)}$ and $G_{2k}^{(1)}$ are the subspaces spanned by the monomials in the e_i's of even and odd degree, respectively.

Theorem 3.5: *Let F be an infinite field. Then:*

1) *A basis of the identities of G_{2k} is given by the polynomials*

$$[x_1, x_2, x_3], \ [x_1, x_2] \cdots [x_{2k+1}, x_{2k+2}]. \tag{9}$$

2)

$$\mathcal{C}(G_{2k}, z) = \exp(z) \sum_{j=0}^{k} \frac{1}{(2j)!} z^{2j}.$$

3)

$$c_n(G_{2k}) = \sum_{j=0}^{k} \binom{n}{2j} \approx \frac{1}{(2k)!} n^{2k}, \quad n \to \infty.$$

Proof: It is easily checked that the polynomials in (9) are identities for G_{2k}.

Let now f be an identity of G_{2k}. As above we may assume that f is a multilinear proper polynomial. After reducing the polynomial f modulo the identities in (9), we obtain that f is a product of Lie commutators of length 2. It can be checked that $[y,x][y,z] \equiv 0$ is a consequence of $[x_1,x_2,x_3] \equiv 0$. Its linearization gives $[y_2,x][y_1,z] \equiv -[y_1,x][y_2,z]$ and this together with $[x_1,x_2,x_3]$ says that the polynomial f can be written as a product of ordered Lie commutators of length 2, i.e., $f = \alpha[x_1,x_2]\cdots[x_{2j+1},x_{2j+2}]$, with $j < k$ (see [6, Theorem 4.1.8]). But an easy substitution proves that f must be the zero polynomial.

From the above it also follows that for $j \leq k$ the polynomial

$$[x_1,x_2]\cdots[x_{2j-1},x_{2j}]$$

is a basis of the multilinear proper polynomials of degree $2j$ mod $\mathrm{Id}(G_{2k})$. Hence $c^p_{2j}(G_{2k}) = 1$ and $c^p_{2j+1}(G_{2k}) = 0$ for $j = 1,\ldots,k$. Hence we get 2) and 3). ∎

A classification of the T-ideals whose codimension growth is at most linear has been carried out in [5]. As a consequence of the above discussion one can classify up to PI-equivalence the PI-algebras with 1 whose sequence of codimensions has at most cubic growth.

Let A be a PI-algebra such that $c_n(A) \approx qn^k$, $k \leq 3$. Since $c^p_0(A) = 1$, $c^p_1(A) = 0$ and $c^p_2(A) \leq 1$, from (2) we obtain that $c_n(A) = 1$ if $k = 0$ and $c_n(A) = 1+n(n-1)/2$ if $k = 2$. Notice that A cannot have linear growth. In case $k = 3$ by the proof of Proposition 2.1, $\chi^p_3(A) = \chi_{(2,1)}$, so $c^p_3(A) = \chi_{(2,1)}(1) = 2$. Since by [4] $c^p_2(A) \neq 0$ we obtain

$$c_n(A) = 1 + \binom{n}{2} + 2\binom{n}{3}.$$

Hence if $k = 0$, $\mathrm{Id}(A) = \mathrm{Id}(F) = \mathrm{Id}(U_1)$. If $k = 2$, since $c^p_3(A) = c^p_4(A) = 0$, we obtain that $[x_1,x_2,x_3] \equiv 0$ and $[x_1,x_2][x_3,x_4] \equiv 0$ are identities of A. Thus $\mathrm{Id}(U_3) \subseteq \mathrm{Id}(A)$ and, since the two algebras have the same growth of the multilinear identities, we get the equality $\mathrm{Id}(U_3) = \mathrm{Id}(A)$.

If $k = 3$, $c^p_4(A) = 0$. Hence $\Gamma_4 \subseteq \mathrm{Id}(A)$ and, since Γ_4 is a basis of the identities of U_4, we obtain that $\mathrm{Id}(U_4) \subseteq \mathrm{Id}(A)$. Since these T-ideals have the same growth also in this case we get $\mathrm{Id}(A) = \mathrm{Id}(U_4)$.

We have proved the following.

Theorem 3.6: *Let A be a unitary algebra over a field F of characteristic zero. If $c_n(A) \leq an^3$, for some $a \geq 1$, then either* $\mathrm{Id}(A) = \mathrm{Id}(F)$ *or* $\mathrm{Id}(A) = \mathrm{Id}(U_3)$ *or* $\mathrm{Id}(A) = \mathrm{Id}(U_4)$.

Unfortunately, a classification of the T-ideals $\mathrm{Id}(A)$ for which $c_n(A) \approx qn^k$, $k \geq 4$ seems to be out of reach at present.

For an algebra A with 1 such that $c_n(A) \approx qn^k$ we know that $r/k! \leq q \leq \sum_{j=2}^{k}(-1)^j/j!$, when $r = 1$ or $r = k-1$ according to whether k is even or odd.

An interesting problem in this setting is to determine all possible values of q for $k \geq 2$. By the discussion before Theorem 3.6 we already know the answer in case $k = 2, 3$. When $k = 4$, $\frac{1}{24} \leq q \leq \frac{9}{24}$ and the S_4-character of Γ_4 has the following decomposition:

$$\chi(\Gamma_4) = \chi_{(3,1)} + \chi_{(2^2)} + \chi_{(2,1^2)} + \chi_{(1^4)}$$

(see [3, Example 12.4.22]). Since $\chi_{(3,1)}(1) = \chi_{(2,1^2)}(1) = 3$, $\chi_{(2^2)}(1) = 2$, $\chi_{(1^4)}(1) = 1$, it is easily seen that q can assume all possible values $q = i/24$, $i = 1, 2, \ldots, 9$. Unfortunately, it is not true in general that q can assume all possible values $q = i/k!$, $i = k-1, k, \ldots, k!(\sum_{j=2}^{k}(-1)^j/j!)$ for k odd. In fact, by considering the decomposition of Γ_5 given in [3, Example 12.4.22], it is possible to check that q cannot take some of the values between $\frac{4}{5!}$ and $\frac{44}{5!}$.

References

[1] V. Drensky, *Codimensions of T-ideals and Hilbert series of relatively free algebras,* Journal of Algebra **91** (1984), 1–17.

[2] V. Drensky, *Relations for the cocharacter sequences of T-ideals,* in *Proceedings of the International Conference on Algebra, Part 2 (Novosibirsk, 1989)*, Contemporary Mathematics, 131, American Mathematical Society, Providence, RI, 1992, pp. 285–300.

[3] V. Drensky, *Free Algebras and PI-Algebras,* Graduate Course in Algebra, Springer-Verlag Singapore, Singapore, 2000.

[4] V. Drensky and A. Regev, *Exact asymptotic behaviour of the codimensions of some P.I. algebras,* Israel Journal of Mathematics **96** (1996), 231–242.

[5] A. Giambruno and D. La Mattina, *PI-algebras with slow codimension growth,* Journal of Algebra **284** (2005), 371–391.

[6] A. Giambruno and M. Zaicev, *Polynomial Identities and Asymptotic Methods,* Mathematical Surveys and Monographs Vol. **122**, American Mathematical Society, Providence, RI, 2005.

[7] I. P. Goulden and D. M. Jackson, *Combinatorial Enumeration,* Wiley, New York, 1983.

[8] A. R. Kemer, *T-ideals with power growth of the codimensions are Specht* (Russian), Sibirskii Matematicheskii Zhurnal **19** (1978), 54–69; English translation: Siberian Mathematical Journal **19** (1978), 37–48.

[9] Ju. N. Mal'cev, *A basis for the identities of the algebra of upper triangular matrices* (Russian), Algebra i Logika **10** (1971), 393–400.

[10] V. M. Petrogradskii, *On complexity functions for T-ideals of associative algebras*, Matematicheskie Zametki **68** no. 6 (2000), 887–897; translation in Mathematical Notes **68** no. 5–6 (2000), 751–759.

[11] A. Regev, *Existence of identities in $A \otimes B$*, Israel Journal of Mathematics **11** (1972), 131–152.

[12] P. N. Siderov, *A basis for identities of an algebra of triangular matrices over an arbitrary field* (Russian), Pliska Studia Mathematica Bulgarica **2** (1981), 143–152.